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Critical state of imbalanced rotating anisotropic disks with small radial and shear moduli

G.G. Portnov [a], M.Y. Ochan [b], C.E. Bakis [c,*]

[a] *Institute of Polymer Mechanics of Latvian University, Riga, 23 Aizkraukles St., Latvia LV-1006, USA*
[b] *Institute of Theoretical Engineering of Russian Academy of Science, 4 Bardin St., Moscow M-117334, Russia*
[c] *Engineering Science and Mechanics, The Pennsylvania State University, 212 Earth-Engineering Sciences Building, University Park, PA 16802, USA*

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Abstract

Critical speeds and the mass center movement of an imbalanced, circumferentially stiff, radially compliant, rotating annular disk on a stiff shaft and bearings are evaluated. It is demonstrated that recently developed hoop-wound composite material disks having elastomeric resin and carbon fibers can enter a critical state before reaching the circumferential strength limit of the material if certain material and geometric relationships are met.

Keywords: Critical state; Critical speed; Anisotropic; Disk; Ring; Composite; Rotating; Elastomeric resin; Flexible resin; Imbalance

1. Introduction

In traditional analyses of the stress–strain state of axisymmetric rotating disks, the volumetric centrifugal force is defined as $\gamma\omega^2 r$, where γ is the density of the material, ω is the angular speed, and r is the radius of a point on a disk in the undeformed condition. Examples of such analyses are reviewed by Timoshenko and Goodier (1970) for isotropic materials and by Lekhnitskii (1968) for anisotropic materials. In accounting for radial displacements (u) in the definition of centrifugal forces, the forces are defined as $\gamma\omega^2(r + u)$ and it is observed that there is a nonproportional relationship between displacement and ω^2. For a certain magnitude of ω^2, stresses and in-plane displacements become unbounded as in the case of critical loads of elastic systems (Timoshenko and Woinowsky-Krieger, 1959; Ziegler, 1968). It appears that the first investigation of such a critical state or "static inertial-elastic instability" in rotating structures considered a uniform, rotating, linear-elastic, isotropic, annular disk on a centric rigid shaft (Brunelle, 1971). Rotating, nonlinear isotropic disks were investigated by Panovko (1985). Tutuncu (2000) investigated rotating, cylindrically orthotropic disks. The latter three publications concerned axisymmetric, in-plane inertial

* Corresponding author. Tel.: +814-865-3178; fax: +814-863-6031.
E-mail address: cbakis@psu.edu (C.E. Bakis).

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loadings and the conditions for unstable displacements. It was suggested that, in the case of low-modulus materials, the instability could become critical prior to the onset of plastic flow.

In kinetic energy storage systems such as flywheels, low-modulus materials have not found application to-date. The development and perfection of flywheels is connected with the use of modern rigid and high-strength carbon fiber composite materials. If rigid polymeric resins are used in these composites, ensuring in the unidirectional composite hoop and radial moduli differing by only one order magnitude, the question of critical speeds for disks comprised of such materials is not of practical concern. However, efforts to suppress dangerous radial tensile stresses that can cause a premature delamination and failure of filament wound disks have resulted in attempts to use highly compliant (elastomeric) resins rather than rigid polymers (Gabrys and Bakis, 1997a). The large radial compliance of a filament wound elastomeric matrix composite disk not only avoids premature delamination, but also leads to a potentially safer failure mode, as the maximal hoop and radial stresses in such disks are limited to a narrow region near the outer radius (Gabrys and Bakis, 1997b).

The critical speeds of a disk that is rigid in the hoop direction and very compliant in shear and in the radial direction was considered for the first time in Ochan (1979), where it was shown that a deviation of inertial loading from axisymmetric could lead to the loss of stability. The work by Ochan (1979) represented a more theoretical rather than practical interest, as composite materials with such a strong difference in properties were absent at the time that paper was written. For conventional composite disks, the critical speed of rotation was not significant; as such speeds exceeded the strength of the disks. The development of new flexible matrix composites has made investigations of the stability of rotation of disks made from such materials rather urgent and requires a wider statement of the problem and a more complete analysis.

The purpose of the present paper is to evaluate the loss of stability during rotation of a disk having large radial and in-plane shear compliances and an initial imbalance causing an asymmetrical distribution of centrifugal loads.

2. Traditional boundary value problem

We first write the system of the governing equations for a plane problem in cylindrical coordinates (r, θ) (see, e.g., Timoshenko and Goodier, 1970), considering a rotating disk as a cylindrically orthotropic body. Setting stresses σ_r, σ_θ, $\tau_{r\theta}$ and displacements in radial (u) and hoop (v) directions as functions of two variables r and θ, one obtains the equations of equilibrium,

$$\begin{aligned} &r\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{r\theta}}{\partial \theta} + (\sigma_r - \sigma_\theta) + rP_r = 0, \\ &\frac{\partial \sigma_\theta}{\partial \theta} + r\frac{\partial \tau_{r\theta}}{\partial r} + 2\tau_{r\theta} + rP_\theta = 0, \end{aligned} \tag{1}$$

where P_r and P_θ are radial and hoop components of volumetric inertial force; the strain–displacement relations,

$$\varepsilon_r = \frac{\partial u}{\partial r}, \quad \varepsilon_\theta = \frac{u}{r} + \frac{\partial v}{r\partial \theta}, \quad \gamma_{r\theta} = \frac{\partial u}{r\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \tag{2}$$

and generalized Hooke's law for a cylindrically orthotropic body,

$$\varepsilon_r = \frac{1}{E_r}(\sigma_r - v_{r\theta}\sigma_\theta), \quad \varepsilon_\theta = \frac{1}{E_\theta}(\sigma_\theta - v_{\theta r}\sigma_r), \quad \gamma_{r\theta} = \frac{\tau_{r\theta}}{G_{r\theta}}, \quad \left(\frac{v_{\theta r}}{E_\theta} = \frac{v_{r\theta}}{E_r}\right), \tag{3}$$

where E_r and E_θ are the moduli of elasticity in the radial and hoop directions, respectively, $v_{r\theta}$ and $v_{\theta r}$ are the Poisson's ratios, and $G_{r\theta}$ is the in-plane shear modulus.

The boundary conditions for internal (a) and external (b) radii of an annular disk, fixed on a rigid shaft, are as follows:

$$\begin{aligned} &u = 0, \quad v = 0 \quad \text{at } r = a, \\ &\sigma_r = 0, \quad \tau_{r\theta} = 0 \quad \text{at } r = b. \end{aligned} \tag{4}$$

3. Body forces in a disk with imbalance

Let us consider a homogeneous, cylindrically orthotropic, uniformly rotating annular disk with an axis of anisotropy coinciding with the main central axis of inertia but displaced relative to the axis of rotation by amount Δ (Δ/r is some small value) due to a slight imbalance (Fig. 1a). The disk is mounted on a relatively rigid shaft and bearing system. It is easy to derive the following ratio between R (the radius in a system of cylindrical coordinates with the center on the axis of rotation) and r (the radius in a cylindrical system of coordinates connected with the center of anisotropy):

$$R^2 = r^2\left[1 + 2\frac{\Delta}{r}\cos\theta + \left(\frac{\Delta}{r}\right)^2\right]. \tag{5}$$

Neglecting here and in the sequel small values of the second order, $(\Delta/r)^2$, one obtains:

$$R \approx r\sqrt{1 + 2\frac{\Delta}{r}\cos\theta} \approx r\left(1 + \frac{\Delta}{r}\cos\theta\right). \tag{6}$$

The corresponding volumetric centrifugal force, directed along radius R and appropriate to the undeformed configuration of a disk is equal to:

$$P_R \approx \gamma\omega^2 r + \gamma\omega^2\Delta\cos\theta, \tag{7}$$

where γ is the mass density of the material of the disk and ω is the angular speed of rotation.

To obtain from P_R the radial (P_r) and hoop (P_θ) components of centrifugal force in the coordinates of cylindrical orthotropy, one determines the angle α using the law of sines (Fig. 1a):

$$\sin\alpha = \frac{\Delta\sin\theta}{R} \approx \frac{\Delta\sin\theta}{r\left(1 + \frac{\Delta}{r}\cos\theta\right)} \approx \frac{\Delta}{r}\left(1 - \frac{\Delta}{r}\cos\theta\right)\sin\theta \approx \frac{\Delta}{r}\sin\theta. \tag{8}$$

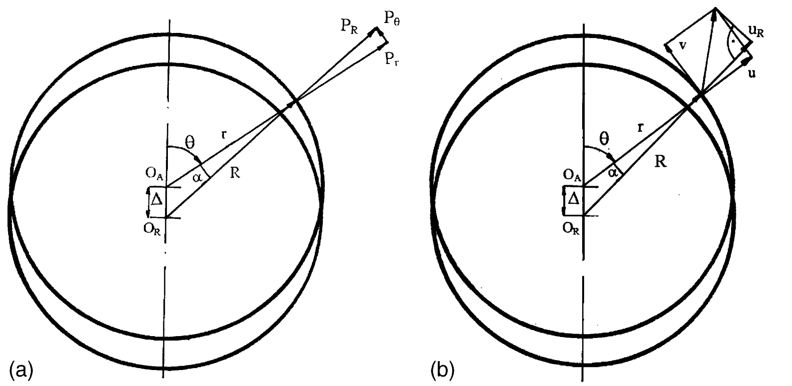

Fig. 1. Inertial forces (a) and displacements (b) in a rotating disk with initial imbalance, Δ, relative to an axis of rotation O_R.

Now it is possible to determine components P_θ and P_r:

$$P_\theta = -P_R \sin\alpha \approx -\gamma\omega^2 r\left(1+\frac{\Delta}{r}\cos\theta\right)\frac{\Delta}{r}\sin\theta \approx -\gamma\omega^2\Delta\sin\theta,$$
$$P_r = P_R\cos\alpha \approx \gamma\omega^2 r\left(1+\frac{\Delta}{r}\cos\theta\right)\sqrt{1-\left(\frac{\Delta}{r}\sin\theta\right)^2} \approx \gamma\omega^2 r + \gamma\omega^2\Delta\cos\theta. \tag{9}$$

As can be seen, at small eccentricity Δ it is possible to assume that $P_r = P_R$.

Linearity of the problem allows the separation of operating loads P_r and P_θ into parts and to consider the action of each of these parts separately. Let us exclude the axisymmetric component of loading (i.e., the term $\gamma\omega^2 r$) and analyze in detail only the influence of nonaxisymmetric loads (i.e., to assume that $P_r = \gamma\omega^2\Delta\cos\theta$ and $P_\theta = -\gamma\omega^2\Delta\sin\theta$). For the consideration of only axisymmetric loads, it is sufficient to take into account the first of the equations of equilibrium (1) together with the reduced set of Eqs. (2) and (3). Further, it is necessary to exclude from all equations shear stresses and deformations as well as hoop displacements. As was previously remarked, such a problem (including u in the definition of loading) was considered by Brunelle (1971) for a linear, isotropic disk and by Tutuncu (2000) for a linear, cylindrically orthotropic disk.

4. Nonaxisymmetric boundary value problem for flexible matrix composite disks

Before departing the traditional boundary value problem to take into account the influence of nonaxisymmetric displacements on centrifugal forces, it is expedient to do some preliminary transformations. Suppose that the nonaxisymmetric parts of loads P_r and P_θ cause stresses and displacements represented as:

$$\sigma_r = \sigma_{r1}(r)\cos\theta, \quad \sigma_\theta = \sigma_{\theta 1}(r)\cos\theta, \quad \tau_{r\theta} = \tau_{r\theta 1}(r)\sin\theta,$$
$$u = u_1(r)\cos\theta, \quad v = v_1(r)\sin\theta. \tag{10}$$

Such a representation allows not only satisfaction of the periodicity condition in θ, but also the separation of variables θ and r in the solution of the governing equations. Substituting the expressions for displacements from (10) into (2), one obtains:

$$\frac{\partial u_1}{\partial r} = \varepsilon_{r1}, \quad v_1 = r\varepsilon_{\theta 1} - u_1, \quad \gamma_{r\theta 1} = -\frac{u_1}{r} + \frac{\partial v_1}{\partial r} - \frac{v_1}{r}. \tag{11}$$

Hooke's law remains in a similar form as in (3):

$$\varepsilon_{r1} = \frac{1}{E_r}(\sigma_{r1} - \nu_{r\theta}\sigma_{\theta 1}), \quad \varepsilon_{\theta 1} = \frac{1}{E_\theta}(\sigma_{\theta 1} - \nu_{\theta r}\sigma_{r1}), \quad \gamma_{r\theta 1} = \frac{\tau_{r\theta 1}}{G_{r\theta}}, \quad \left(\frac{\nu_{\theta r}}{E_\theta} = \frac{\nu_{r\theta}}{E_r}\right). \tag{12}$$

Boundary conditions (4) are transformed to the form:

$$\sigma_{r1}(b) = \tau_{r\theta 1}(b) = u_1(a) = v_1(a) = 0. \tag{13}$$

In (11) and (12) all unknown displacements, strains, and stresses are functions of r, as in (10).

We next assume that the material of a disk is rigid in the hoop direction ($E_\theta = \infty$). Such an assumption completely excludes the possibility of a critical state (loss of stability) of an axisymmetric form and is quite acceptable for the description of a composite based on carbon fibers in a flexible polyurethane resin (Gabrys and Bakis, 1997b), where $E_\theta/E_r \approx 1700$. For $E_\theta = \infty$, it follows from the fourth relation in (12) that $\nu_{r\theta} = 0$, and from the second relation in (12) that $\varepsilon_{\theta 1} = 0$. By substituting $\varepsilon_{\theta 1} = 0$ into (11), one obtains

$$v_1 = -u_1, \qquad \varepsilon_{r1} = -\gamma_{r\theta 1}. \tag{14}$$

After substitution of Poisson's ratio $v_{r\theta} = 0$ into the expression for ε_{r1} in (12), one obtains, instead of (12), the following relations:

$$\frac{1}{G_{r\theta}}\tau_{r\theta 1} = -\frac{1}{E_r}\sigma_{r1}, \qquad \frac{\partial u_1}{\partial r} = \frac{1}{E_r}\sigma_{r1}. \tag{15}$$

For determination of the increment of centrifugal force R_R due to the nonaxisymmetric displacements, we first calculate displacement u_R in a direction R as a projection to this direction of displacements u and v in the axes of cylindrical orthotropy (Fig. 1b). Using relations from (10) and (14), one obtains:

$$u_R = u\cos\alpha - v\sin\alpha = u_1\cos\theta\cos\alpha - v_1\sin\theta\sin\alpha. \tag{16}$$

Taking into account (8) it is possible with sufficient accuracy to accept that $u_R = u_1\cos\theta = u$.

Increments of centrifugal forces δP_R in a direction R and their components δP_r and δP_θ in directions r and θ, respectively, are given by the following relations:

$$\begin{aligned} &\delta P_R = \gamma\omega^2 u_1\cos\theta, \\ &\delta P_r = \gamma\omega^2 u_1\cos\theta\cos\alpha \approx \gamma\omega^2 u_1\cos\theta, \\ &\delta P_\theta = -\gamma\omega^2 u_1\cos\theta\sin\alpha \approx -\gamma\omega^2 u_1\frac{\Delta}{r}\cos\theta\sin\theta. \end{aligned} \tag{17}$$

Substituting the calculated values of volumetric inertial loads from (9) (excluding axisymmetric component) and their increments from (17) into (1) leads to the equations of equilibrium for the nonaxisymmetric part of the problem:

$$\begin{aligned} &r\frac{\partial\sigma_r}{\partial r} + \frac{\partial\tau_{r\theta}}{\partial\theta} + (\sigma_r - \sigma_\theta) = -\gamma\omega^2 r\Delta\cos\theta - \gamma\omega^2 u_1 r\cos\theta, \\ &r\frac{\partial\tau_{r\theta}}{\partial r} + \frac{\partial\sigma_\theta}{\partial\theta} + 2\tau_{r\theta} = \gamma\omega^2 r\Delta\sin\theta + \gamma\omega^2 u_1\Delta\cos\theta\sin\theta. \end{aligned} \tag{18}$$

Substituting in (18) expressions for stresses from (10) reveals that all terms contain $\cos\theta$ in the first equation and $\sin\theta$ in the second ($\cos\theta$ in the second equation will be excluded later). After simplification, one obtains (prime designates differentiation with respect to r):

$$\begin{aligned} &r\sigma'_{r1} + \tau_{r\theta} + (\sigma_{r1} - \sigma_{\theta 1}) = -\gamma\omega^2 r\Delta - \gamma\omega^2 u_1 r, \\ &r\tau'_{r\theta 1} - \sigma_{\theta 1} + 2\tau_{r\theta 1} = \gamma\omega^2 r\Delta + \gamma\omega^2 u_1\Delta\cos\theta. \end{aligned} \tag{19}$$

After transformations with use of relations (14), one obtains:

$$ru_1''(E_r + G_{r\theta}) + u_1'(E_r + G_{r\theta}) = -\gamma\omega^2 u_1 r\left(1 + \frac{\Delta}{r}\cos\theta\right) - 2\gamma\omega^2 r\Delta. \tag{20}$$

Since $\Delta/r \ll 1$, it is possible to exclude the term containing $\cos\theta$ from (20) and to express this equation as:

$$(ru_1')' + \lambda^2 r u_1 = -2\lambda^2\Delta r, \tag{21}$$

where $\lambda^2 = \gamma\omega^2/(E_r + G_{r\theta})$.

The boundary conditions for the problem according to (13) and (15) are:

$$u_1(a) = 0, \qquad u_1'(b) = 0. \tag{22}$$

It is necessary to note that satisfaction of the two conditions (22) means automatic satisfaction of all four conditions (13): from $u_1(a) = 0$ and (14) it follows that $v_1(a) = 0$; from $u_1'(b) = 0$ and (15) it follows that $\sigma_{r1}(b) = 0$, and $\tau_{r\theta 1}(b) = 0$.

Eq. (21) with homogeneous boundary conditions (22) is the Sturm–Liouville problem (Bessel's equation). Its solution, satisfying the given boundary conditions at fixed values of λ, can be expressed in the form of series of normalized eigenfunctions φ_n and corresponding eigenvalues λ_n (Levitan, 1950):

$$u_1 = \sum_{n=0}^{\infty} \frac{C_n}{\lambda^2 - \lambda_n^2} \varphi_n(r), \qquad C_n = \int_a^b \varphi_n(r) f(r)\,\mathrm{d}r, \tag{23}$$

where $f(r) = -2\lambda^2 \Delta r$ is the right-hand side of Eq. (21).

As it is visible from (23), at $\lambda = \lambda_n$, displacements tend to infinity. Values of ω corresponding to these eigenvalues are the critical speeds of the disk. Critical states correspond to values of λ at which there are nontrivial solutions to the homogeneous part of the governing equation: $(ru_1')' + \lambda^2 r u_1 = 0$; $u_1(a) = 0$, $u_1'(b) = 0$. Thus, the investigation of critical states of a rotating disk amounts to a search for eigenvalues of the homogeneous part of the governing equation (21).

We cast the governing equation (21) in a dimensionless form, substituting $\rho = (r/b)(m = a/b \leqslant \rho \leqslant 1)$, $\bar{u}_1 = u_1/b$, and $\overline{\Delta} = \Delta/b$. Then, Eq. (21) can be written as follows:

$$(\rho \bar{u}_1')' + \bar{\lambda}^2 \rho \bar{u}_1 = -2\bar{\lambda}^2 \overline{\Delta} \rho, \tag{24}$$

where $\bar{\lambda}^2 = \gamma \omega^2 b^2/(E_r + G_{r\theta})$ is a dimensionless parameter. The boundary conditions are:

$$\bar{u}_1(m) = 0, \qquad \bar{u}_1'(1) = 0. \tag{25}$$

It is necessary to note that the limiting value of ωb in $\bar{\lambda}^2$—the peripheral speed on a disk—is determined by strength of its material and does not depend on the absolute size of a disk. For a disk comprised of modern high-strength composites, the limiting value of ωb is approximately 1000–1500 m/s. This upper limit of ωb determines the upper limit of $\bar{\lambda}^2$, above which the search for critical states of a rotating disk caused by its radial and shear compliances is not of practical interest. For example, for the carbon fiber composite with polyurethane resin AS4C/PET80 described in Gabrys and Bakis (1997b), $\gamma = 1460$ kg/m^3 and $E_r + G_{r\theta} \approx 0.1$ GPa. The limiting speed of rotation can be estimated based on hoop strength, S_θ, which for the given material is equal to 2180 MPa: $\omega b = \sqrt{S_\theta/\gamma} \approx 1200$ m/s. These data allow the estimation of the upper limit of parameter $\bar{\lambda}_{\lim}$ corresponding to failure of the disk from hoop stresses: $\bar{\lambda}_{\lim} \approx 4.6$. At the same time, use of the same fibers with epoxy resin (carbon composite AS4C/Epon 9405 epoxy, also described in (Gabrys and Bakis, 1997b), with $\gamma = 1554$ kg/m^3, $E_r + G_{r\theta} \approx 13.9$ GPa, $S_\theta = 2650$ MPa) results in $\bar{\lambda}_{\lim} \approx 0.437$.

We next turn to the determination of eigenvalues of the homogeneous part of the governing equation with appropriate boundary conditions (24) and (25):

$$(\rho \bar{u}_1')' + \bar{\lambda}^2 \rho \bar{u}_1 = 0, \tag{26}$$

$$\bar{u}_1(m) = \bar{u}_1'(1) = 0. \tag{27}$$

The general solution for this equation (see, e.g., Kamke, 1971) has the form:

$$\bar{u}_1 = A J_0(\bar{\lambda}\rho) + B Y_0(\bar{\lambda}\rho), \tag{28}$$

where J_0 and Y_0 are zero order Bessel functions of the first and second kind and A and B are constants. The eigenvalues are determined from a condition of existence of nontrivial solutions for Eq. (26). Such condition is the equality to zero of the determinant of system of the equations for determination of constants from boundary conditions (27):

$$J_0(\bar{\lambda}m) Y_1(\bar{\lambda}) - J_1(\bar{\lambda}) Y_0(\bar{\lambda}m) = 0. \tag{29}$$

The first eigenvalues, $\bar{\lambda}_1$, for various normalized sizes of a disk m, are shown in Fig. 2. As may be seen, the danger of occurrence of a critical state (instability) in a disk of flexible matrix composite with $\bar{\lambda}_{\lim} = 4.6$ is

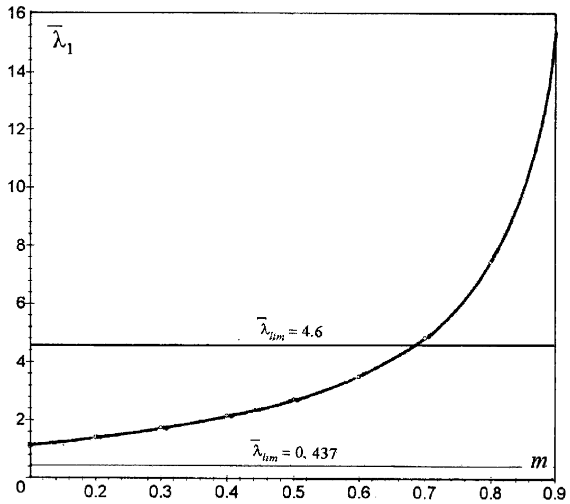

Fig. 2. Dependence of the first eigenvalues $\bar{\lambda}_1$ upon the relative sizes of the disks, m.

quite real up to $m = 0.686$. For a stiff disk, even at $m = 0.1$, the critical speed is more than twice the limiting speed based on circumferential strength. Given knowledge of the value of $E_r + G_{r\theta}$, Fig. 2 allows the estimation of the occurrence of a critical state in a linear-elastic radially compliant disk of any size.

The solution of Eq. (24) for cases where $\bar{\lambda}$ is not equal to an eigenvalue of the problems (24) and (25) can be represented as the sum of the general solution (28) of the homogeneous equation and the particular solution of the nonhomogeneous equation, $-2\overline{\Delta}$:

$$\bar{u}_1(\bar{\lambda}, \rho) = AJ_0(\bar{\lambda}\rho) + BY_0(\bar{\lambda}\rho) - 2\overline{\Delta}. \tag{30}$$

Using boundary conditions (25) for determination of constants A and B, one obtains:

$$\bar{u}_1(\bar{\lambda}, \rho) = 2\overline{\Delta}\left(\frac{J_0(\bar{\lambda}\rho)Y_1(\bar{\lambda}) - J_1(\bar{\lambda})Y_0(\bar{\lambda}\rho)}{J_0(\bar{\lambda}m)Y_1(\bar{\lambda}) - J_1(\bar{\lambda})Y_0(\bar{\lambda}m)} - 1\right). \tag{31}$$

Hoop elements in this case do not change shape—i.e., they remain circles and are only displaced along the radial direction $\theta = 0$. To demonstrate this result, consider the displacement d of a point on the hoop element:

$$d = u\cos\theta - v\sin\theta, \tag{32}$$

where u is the radial displacement of the point and v is the hoop displacement. From (10) and the first of Eq. (14) one has:

$$d = u_1\cos^2\theta - v_1\sin^2\theta = u_1(\cos^2\theta + \sin^2\theta) = u_1. \tag{33}$$

Thus, the in-plane displacement, d, of a point on the hoop element does not depend on coordinate θ and, hence, is constant for all points on the hoop element. In other words, concentric circles on the disk are displaced and do not change shape. This circumstance enables a rather simple determination of the displacement of the center of mass of a rotating homogeneous disk (Δ_{cm}) with mass $M = \gamma\pi(b^2 - a^2)H$, where H is the axial thickness of disk and Δ is the initial imbalance. Normalizing all linear distances to the outside radius, b, and presuming a constant density of the disk, one obtains for $\overline{\Delta}_{cm} = \Delta_{cm}/b$, using integration of displacement of the centers of mass of hoop differential elements, the following expression:

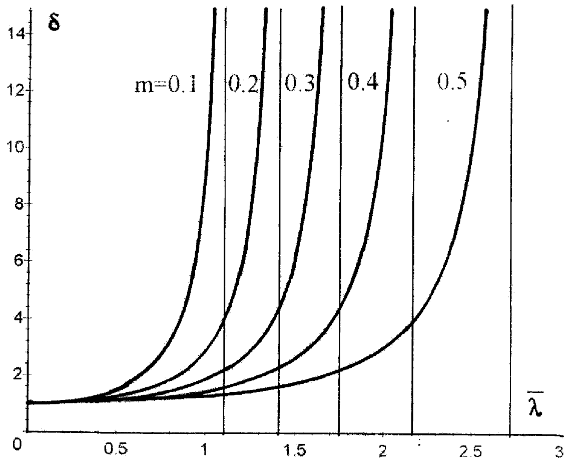

Fig. 3. Dependence of the normalized relative displacement of the center of mass, δ, upon the parameter $\bar{\lambda}$ approaching critical speeds in disks of various relative sizes, m.

$$\overline{\varDelta}_{\mathrm{cm}}(\lambda,\rho)=\overline{\varDelta}+\frac{2}{1-m^2}\int_m^1\bar{u}_1(\bar{\lambda},\rho)\rho\,\mathrm{d}\rho$$
$$=\overline{\varDelta}\left[1+\frac{4}{1-m^2}\left(\frac{(J_1(\bar{\lambda})-mJ_1(\bar{\lambda}m))Y_1(\bar{\lambda})-(Y_1(\bar{\lambda})-mY_1(\bar{\lambda}m))J_1(\bar{\lambda})}{\bar{\lambda}(J_0(\bar{\lambda}m)Y_1(\bar{\lambda})-J_1(\bar{\lambda})Y_0(\bar{\lambda}m))}-\frac{1-m^2}{2}\right)\right]. \quad (34)$$

By this means, the displacement of the center of mass occurs due to the displacements of hoop elements during the loss of stability of the disk. The dependence of the normalized relative displacement of the mass center, $\delta=\overline{\varDelta}_{\mathrm{cm}}/\overline{\varDelta}$, on $\bar{\lambda}$ for disks of various normalized sizes, m, is represented in Fig. 3. Eq. (34) is approximated very precisely up to $\bar{\lambda}=\bar{\lambda}_1$ by the equation for δ of a rigid disk on an elastic shaft:

$$\delta=1+\frac{k\bar{\lambda}^2}{\bar{\lambda}_1^2(m)-\bar{\lambda}^2}, \quad (35)$$

where $\bar{\lambda}_1$ is the first eigenvalue for a given m and k is a coefficient depending upon m and linearily varying from 0.9 to 1.6 as m changes from 0.1 to 0.9. Strictly speaking, in accordance with the initial assumptions of our problem, relations (34) and (35) are correct only for rather small $\overline{\varDelta}_{\mathrm{cm}}$.

5. Closure

The solution for critical speeds and mass center movement of a circumferentially stiff, radially compliant disk on a rigid shaft and bearings resembles the solution for a rotating rigid disk on a flexible shaft. However, such similarity exists only in theory. It is important to note that, in our case, there is no opportunity for self-centering of the disk. As is known (Panovko, 1985; Den Hartog, 1956), self-centering of a rotor with a flexible shaft is carried out under the influence of Coriolis acceleration, which occurs as soon as the center of mass of a disk begins to move in a radial direction from the center of rotation. In such a case, the center of mass undergoes movement in the tangential direction and eventually is located at the other side of the center of rotation. During this process, the collinearity of points describing the positions of the bearing axis, the center of the shaft, and the center of mass of a disk is broken. As noted in Panovko (1985) and Den Hartog (1956), if such tangential movement of the mass center is somehow prevented, the rotor appears unstable at speeds above the critical speed. Such is indeed the situation in the considered problem. Coriolis acceleration will lead only to a change of speed of rotation of the rigid shaft in rigid bearings and the self-centering will not be realized. The situation is close to that described in Panovko (1985) and Den Hartog (1956) for the case of constant-speed rotation about a vertical axis of a system consisting of a massless framework and an attached rectilinear wire on which a mass, supported by a spring, can move without friction. At zero angular speed, the mass is displaced relative to the axis of rotation. The movement of the mass occurs only in the radial direction and the position of the mass at speeds greater than the critical speed is unstable.

Thus, a circumferentially stiff, radially compliant disk on a rigid shaft rotating in rigid bearings becomes unstable upon reaching a critical speed. Stable configurations cannot be found at higher speeds. A means of prevention of a critical state is, obviously, the installation of the shaft in bearings with small rigidity. However, the problem of a rotating flexible shaft/bearing system with a radially compliant disk, in which imbalance grows with increasing of speed of rotation, requires separate consideration.

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